Total marks – 84
Attempt Questions 1 – 7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) **Marks**

(a) Solve $\dfrac{4}{x+1} \geq 1$ **3**

(b) Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ **2**

(c) Let A be the point (-2, 1) and B be the point (5, 2). **2**
Find the coordinates of the point P which divides AB externally in the ratio 5:4

(d) Indicate on a number plane the region satisfied by both $x^2 + y^2 \leq 1$ and $y \geq x$ **2**

(e) Use the substitution $u^2 = x - 2$ to evaluate $\displaystyle\int_2^3 x\sqrt{x-2}\,dx$ **3**

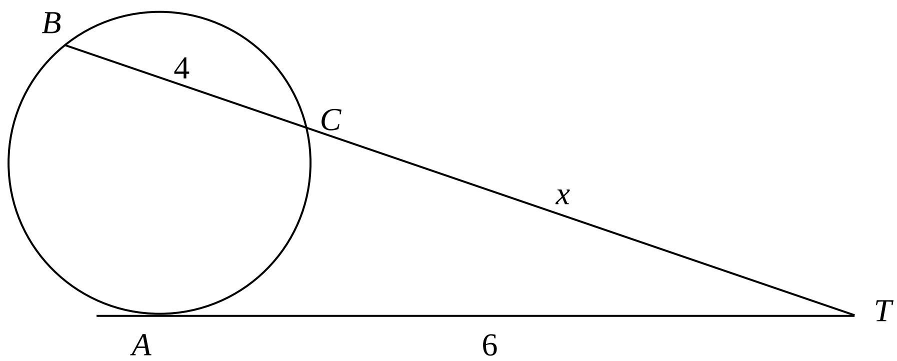
Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The polynomial equation $2x^3 - 3x^2 + 4x - 7 = 0$ has roots α, β and γ. **3**

Find the exact value of $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$.

(b) Find $\frac{d}{dx}\cos^{-1}\left(4x^3\right)$. **2**

(c) **2**



The line *AT* is the tangent to the circle at *A*, and *BT* is a secant meeting the circle at *B* and *C*.

Given that *AT* = 6, *BC* = 4 and *CT* = *x*, find the exact value of *x*.

(d) Use the *t*-method to solve $5\sin x - 2\cos x = 2$, where $0° \le x \le 360°$ **3**

(e) A debating team of three people is to be chosen from five English teachers and four Mathematics teachers.

(i) In how many ways can the team be chosen? **1**

(ii) What is the probability that that the whole of the team will be Mathematics teachers? **1**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \cos^2 6x \, dx$ **2**

(b) Let $P(x) = 3x^3 - ax^2 - bx + 1$, where $P(x)$ is a polynomial and a and b are real numbers. **3**
When $P(x)$ is divided by $(x - 1)$ there is no remainder.
When $P(x)$ is divided by $(x + 2)$ the remainder is 15.

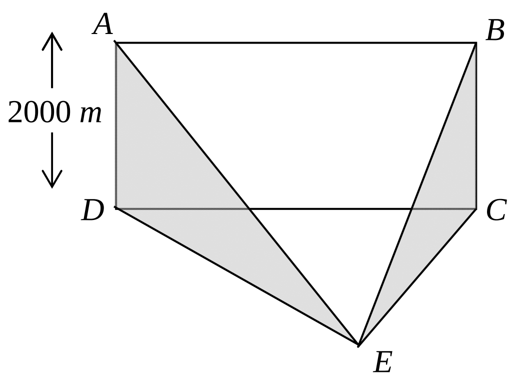
Find the values of a and b.

(c) Two parallel lines have equations $x - my + 1 = 0$ and $x - my - 1 = 0$

(i) Show that $\left(0, \frac{1}{m}\right)$ lies on $x - my + 1 = 0$. **1**

(ii) Find m given that the perpendicular distance between the lines is 1 unit. **2**

(d)



A plane is flying horizontally from A to B at a height of 2000m. **4**
From a point E the angle of elevation of A is 15º.
From E the angle of elevation of B is 25º.

If $\angle DEC = 60°$, calculate the distance AB to the nearest metre.

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use mathematical induction to prove that **3**

$$\frac{1}{1\times 5}+\frac{1}{5\times 9}+\ldots+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}$$

(b) The point $P\left(2ap, ap^2\right)$ lies on the parabola $x^2 = 4ay$.

(i) Prove that the tangent at P cuts the y-axis at $T\left(0, -ap^2\right)$. **2**

(ii) M is the midpoint of the interval PT. **2**
Find the locus of M.

(c) Maddison is one of ten members of the chess club. Each week one member is selected at random to win a prize.

(i) What is the probability that in the first 7 weeks Maddison will win at least 1 prize? **1**

(ii) Show that in the first 20 weeks, Maddison has a greater chance of winning exactly two prizes, than of winning exactly one prize. **2**

(iii) For how many weeks must Maddison participate in the prize drawing so that she has a greater chance of winning exactly 3 prizes than of winning exactly 2 prizes. **2**

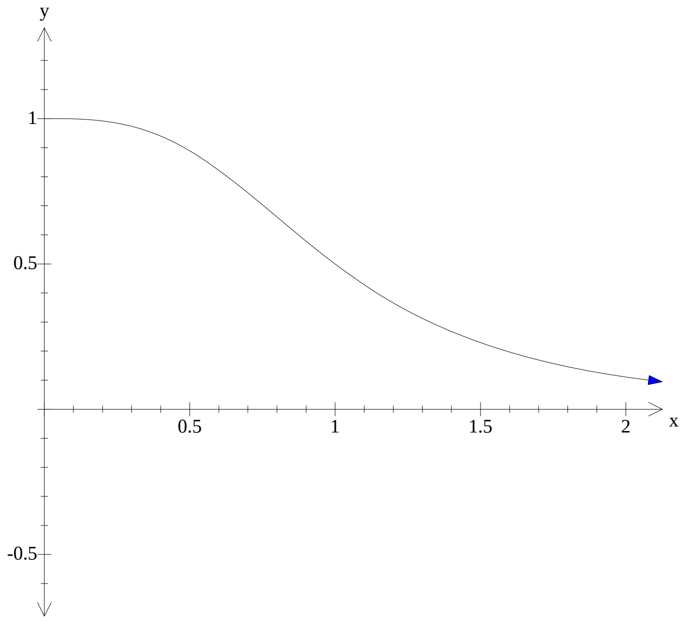
Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Evaluate $\int_2^8 \frac{8}{x}\,dx$, write your answer in the form $a\log 2$, where a is an integer **2**

(ii) Use Simpson's Rule with 3 function values to find an approximation for $\int_2^8 \frac{8}{x}\,dx$ **2**

(iii) Hence find an approximation for log 2, correct to 3 decimal places. **1**

(b) The diagram below shows a sketch of the graph $y = f(x)$, where $f(x) = \frac{1}{1+x^3}$,
for $x \geq 0$.



(i) Copy or trace this diagram into your answers. **1**
On the same set of axes, sketch the inverse function, $y = f^{-1}(x)$.

(ii) State the domain of $f^{-1}(x)$. **1**

(iii) Find an expression for $f^{-1}(x)$ in terms of *x*. **2**

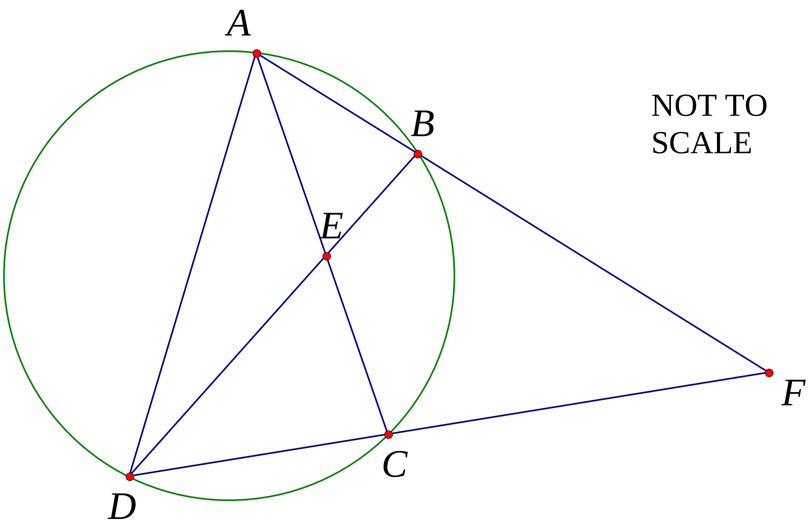
(iv) The graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at exactly one point *P*. **1**
Let α be the *x*-coordinate of *P*. Explain why α is a root of the equation

$$x^4 + x - 1 = 0.$$

(v) Take 0.5 as a first approximation to α. Use one application of Newton's method to find a second approximation for α. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The points A, B, C and D are placed on a circle of radius r such that AC and BD meet at E. The lines AB and DC are produced to meet at F, and $BECF$ is a cyclic quadrilateral.

Copy or trace this diagram into your answer booklet.

(i) Find the size of $\angle DBF$, giving reasons for your answer. **2**

(ii) Find an expression for the length of AD in terms of r. **2**

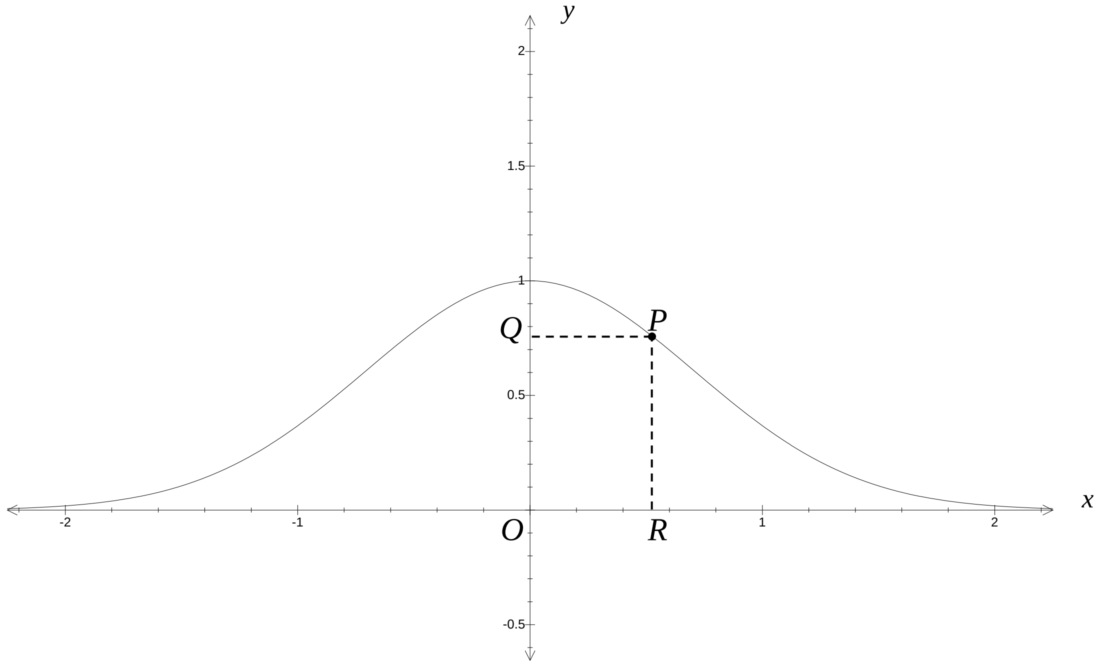
(b) In the expansion of $(1+ax)^9$ the coefficient of x^5 is twice the coefficient of x^6. Find the value of the constant a. **4**

(c) A die is biased so that in any single throw the probability of an odd score is p, where p is a constant such that $0 < p < 1,\ p \neq \frac{1}{2}$.

(i) Show that in six throws of the die the probability of at most one even score is given by $6p^5 - 5p^6$. **2**

(ii) Find the probability that in six throws of the die the product of the scores is even. **2**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**



(a) P is a point on the curve $y = e^{-x^2}$. The origin is at O and perpendiculars through P to the coordinate axes meet those axes at Q and R. Show that the maximum area of the rectangle $OQPR$ is given by $\frac{1}{\sqrt{2e}}$. **5**

(b) (i) Show that $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$. **2**

(ii) Prove that the derivative of $y = e^x \sin x$ is given by $\frac{dy}{dx} = \sqrt{2}e^x \sin\left(x + \frac{\pi}{4}\right)$ **2**

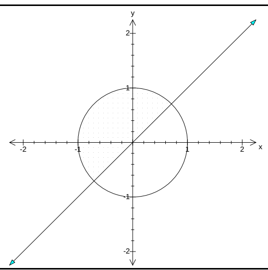
(iii) If $y = e^x \sin x$ prove by mathematical induction that the n^{th} derivative is given by $\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin\left(x + \frac{n\pi}{4}\right)$. **3**

HSC Extension 1 Mathematics Internal Examination Solutions 2005

Question	Criteria	Marks	Bands
1(a)	$\frac{4}{x+1} \geq 1, x \neq -1,$	1	
	Let $\frac{4}{x+1} = 1$		
	$x = 3$	1	
	$-1 \leq x \leq 3$	1	

Question	Criteria	Marks	Bands
1(b)	$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^3$	1	
	$= \frac{\pi}{2}$	1	

Question	Criteria	Marks	Bands
1 (c)	(-2, 1) (5, 2). 5:-4		
	$x = \frac{kx_2 + lx_1}{k+l}, y = \frac{ky_2 + ly_1}{k+l}$		
	$x = \frac{5\times5+-4\times-2}{5+-4}, y = \frac{5\times2+-4\times1}{5+-4}$	1	
	$x = 33, y = 6 \rightarrow P(33, 6)$	1	

Question	Criteria	Marks	Bands
1(d)	 1 for correct graphs	1	
	1 for correct shading	1	

Question	Criteria	Marks	Bands
1(e)	$\int_2^3 x\sqrt{x-2}\,dx \quad x-2=u^2 \rightarrow x=u^2+2 \rightarrow dx = 2udu$	1	
	$u_1^2 = 3-2 \rightarrow u_1 = 1, u_1^2 = 2-2 \rightarrow u_2 = 0$		
	$\int_2^3 x\sqrt{x-2}\,dx = \int_0^1 \left(u^2+2\right)\sqrt{u^2}\,2udu$	1	
	$2\int_0^1 \left(u^4+2u^2\right)du = 2\left[\frac{u^5}{5}+\frac{2u^3}{3}\right]_0^1 = \frac{26}{15}$	1	

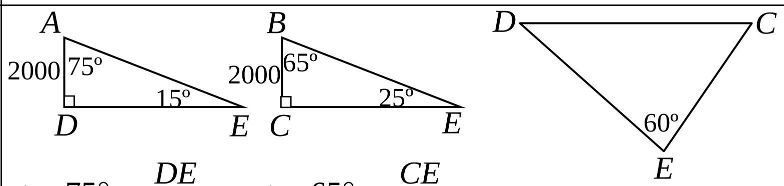
Question	Criteria	Marks	Bands
2(a)	$2x^3-3x^2+4x-7=0$ $\alpha+\beta+\gamma=\frac{3}{2},\alpha\beta+\alpha\gamma+\beta\gamma=2,\alpha\beta\gamma=\frac{7}{2}$ $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}=\frac{\gamma+\beta+\alpha}{\alpha\beta\gamma}$ $=\frac{3}{2}\div\frac{7}{2}=\frac{3}{7}$	 1 1 1	

Question	Criteria	Marks	Bands
2(b)	$\frac{d}{dx}\cos^{-1}\left(4x^3\right)=\frac{-1}{\sqrt{1-\left(4x^3\right)^2}}\times12x^2$ one mark if only the denominator is correct $=\frac{-12}{\sqrt{1-16x^6}}$	1 1	

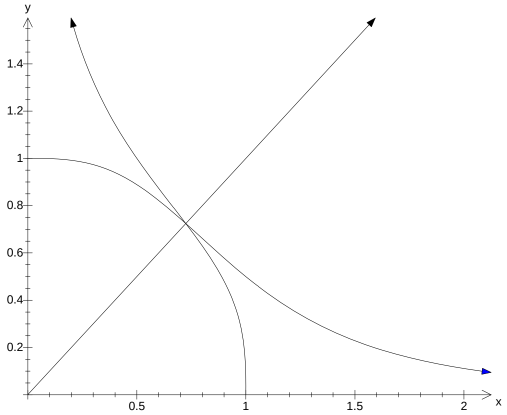
Question	Criteria	Marks	Bands
2(c)	$x\left(x+4\right)=6^2$(product of secants = square of the tangent) $x^2+4x=36$ $\left(x+2\right)^2=40$ $x=-2+2\sqrt{10}$	1 1	

Question	Criteria	Marks	Bands
2(d)	$5\sin x-2\cos x=2\to5\times\frac{2t}{1+t^2}-2\times\frac{1-t^2}{1+t^2}=2$ $10t-2+2t^2=2+2t^2$ $10t=4\to t=0.4\to\tan\frac{x}{2}=0.4$ $\frac{x}{2}=21°48',x=43°36'$ check 180º; 5 sin 180º − 2 cos180º=2✓ $x=43°36',180°$	1 1 1	

Question	Criteria	Marks	Bands
2(e)(i)	${}^9C_3=84$	1	
2(e)(ii)	$\frac{{}^4C_3}{{}^9C_3}=\frac{4}{84}=\frac{1}{21}$	1	

Question	Criteria	Marks	Bands
3(a)	$\cos^2 x = \frac{1}{2}(\cos 2x + 1) \to \cos^2 6x = \frac{1}{2}(\cos 12x + 1)$ $\int \cos^2 6x\, dx = \int \frac{1}{2}(\cos 12x + 1)\, dx = \frac{\sin 12x}{24} + \frac{x}{2} + C$	1 1	
3(b)	$P(x) = 3x^3 - ax^2 - bx + 1$ $P(1) = 0 \to 3 - a - b + 1 = 0 \to a + b = 4$ $P(-2) = 15 \to -24 - 4a - 2b + 1 = 15 \to -2a + b = 19$ $-2a + b = 19$ $\underline{a + b = 4}$ $a = -5, b = 9$	1 1 1	
3(c)(i)	$x - my + 1 = 0$ $0 - m\left(\frac{1}{m}\right) + 1 = 0$, $\left(0, \frac{1}{m}\right)$ $0 = 0$ ✓	1	
3(c)(ii)	$x - my - 1 = 0$, $\left(0, \frac{1}{m}\right)$ $p.d. = \frac{\lvert ax_1 + by_1 + c \rvert}{\sqrt{a^2 + b^2}}$ $x - my - 1 = 0, \left(0, \frac{1}{m}\right)$ $p.d. = \frac{\lvert 0 - 1 - 1 \rvert}{\sqrt{1^2 + (-m)^2}}$ $= \frac{2}{\sqrt{1 + m^2}} = 1 \to m = \pm\sqrt{3}$	1 1	
3(d)	 $\tan 75° = \frac{DE}{2000}$ $\tan 65° = \frac{CE}{2000}$ now *DC* = *AB* $DE = 2000 \tan 75°$ $CE = 2000 \tan 65°$ $DC^2 = (2000 \tan 65°)^2 + (2000 \tan 75°)^2$ $-2 \times 2000 \tan 65° \times 2000 \tan 75° \cos 60°$ *DC* = 6488m (nearest metre) (1 mark for diagrams, 1 for relationships for *DE* and *CE*)	1 1 1 1	

Question	Criteria	Marks	Bands
4(a)	$\frac{1}{1\times5}+\frac{1}{5\times9}+\ldots+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}$ Test n = 1: LHS = $\frac{1}{1\times5}=\frac{1}{5}$ RHS = $\frac{1}{4\times1+1}=\frac{1}{5}$ true for n=1 Suppose its true for some value of n, say k. $\frac{1}{1\times5}+\frac{1}{5\times9}+\ldots+\frac{1}{(4k-3)(4k+1)}=\frac{k}{4k+1}$ Prove true for $n = k + 1$; RTP: $\frac{1}{1\times5}+\frac{1}{5\times9}+\ldots+\frac{1}{(4k-3)(4k+1)}+\frac{1}{(4k+1)(4k+5)}=\frac{k+1}{4k+5}$ $LHS=\frac{k}{4k+1}+\frac{1}{(4k+1)(4k+5)}$ $=\frac{k(4k+5)+1}{(4k+1)(4k+5)}$ $=\frac{4k^2+5k+1}{(4k+1)(4k+5)}=\frac{(4k+1)(k+1)}{(4k+1)(4k+5)}$ $=\frac{k+1}{4k+5}$ Proved by induction	1 1 1	
4(b)(i)	$x^2=4ay\to y=\frac{x^2}{4a}\to\frac{dy}{dx}=\frac{2x}{4a}\left(2ap,ap^2\right)$ $m=\frac{2\times2ap}{4a}=p\to y-ap^2=p(x-2a)\to y=px-ap^2$ cuts the y-axis when $x = 0\to y=-ap^2\to T=\left(0,-ap^2\right)$	1 1	
4(b)(ii)	$M=\left(\frac{2ap+0}{2},\frac{-ap^2+ap^2}{2}\right)=\left(ap,0\right)$ The locus of M is $y = 0$.	1 1	
4(c)(i)	P(at least one) = 1 – P(none) P(at least one) = 1 – $\left(\frac{9}{10}\right)^7$ = 0.52 (2dp)	1	
4(c)(ii) 4(c) (iii)	$P(2\text{ wins})={}^{20}C_2\left(\frac{9}{10}\right)^{18}\left(\frac{1}{10}\right)^2=0.285\left(\text{3dp}\right)$ $P(1\text{win})\,{}^{20}C_1\left(\frac{9}{10}\right)^{19}\left(\frac{1}{10}\right)^1=0.270\left(\text{3dp}\right)$ $\therefore P(2\text{ wins})>P(1\text{win})$ n = 30	1 1	

Question	Criteria	Marks	Bands
5(a)(i)	$\int_2^8 \frac{8}{x}\,dx = 8\left[\ln x\right]_2^8$ $= 8\left(\ln 8 - \ln 2\right)$ $= 8\ln 4 = 8\ln 2^2$ $= 16\ln 2$	1 1	
5(a)(ii)	$\int_2^8 \frac{8}{x}\,dx \approx \frac{8-2}{6}\left(f(2)+4f(5)+f(8)\right)$ $= 4 + 4\times 1.6 + 1$ $= 11.4$	1 1	
5(a)(iii)	16 ln 2$16\ln 2 \approx 11.4 \rightarrow \ln 2 \approx \frac{11.4}{16} = 0.7125$	1	
5(b)(i)		1	
5(b)(ii)	$x \le 1$	1	
5(b)(iii)	$y = \frac{1}{1+x^3}$ To find the inverse function swap *x* and *y*. $x = \frac{1}{1+y^3}$ $1 + y^3 = \frac{1}{x}$ $y^3 = \frac{1}{x} - 1$ $y = \sqrt[3]{\frac{1-x}{x}}$	1 1	
5(b)(iv)	A function and its inverse intersect on the line $y = x$ $\frac{1}{1+x^3} = x \rightarrow 1 = x + x^4 \rightarrow x^4 + x - 1 = 0$	1	
5(b)(v)	$f(x) = x^4 + x - 1,\ f'(x) = 4x^3 + 1 \rightarrow f\left(\tfrac{1}{2}\right) = \tfrac{-7}{16} \rightarrow f'\left(\tfrac{1}{2}\right) = \tfrac{3}{2}$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{2} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)} = \frac{1}{2} - \frac{\frac{-7}{16}}{\frac{3}{2}} = \frac{19}{24}$	1 1	

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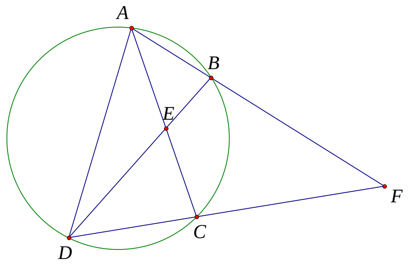
<tr><th>Question</th><th>Criteria</th><th>Marks</th><th>Bands</th></tr>
<tr><td>6(a)</td><td>

Let $\angle DBF = x \to \angle ECF = 180° - x$ ($EBFC$ is cyclic).
$\angle ECD = x$ (straight angle)
$\angle ABD = x$ (subtended by arc AD)
$but\ \angle EBF + \angle ABD = 180° \to 2x = 180° \to x = 90°$</td><td>1
1</td><td></td></tr>
<tr><td>6(a)(ii)</td><td>From (i) $\angle DBF = 90° \to \angle ABD = 90°$ (straight angle)
This means that $\angle ABD$ is the angle in a semi-circle.
So AD is a diameter $\to$ length of $AD = 2r$.</td><td>1
1</td><td></td></tr>
<tr><td>6(b)</td><td>The coefficient of x^5 = ${}^9C_5 a^5 = 126a^5$
The coefficient of x^6 = ${}^9C_6 a^6 = 84a^6$
$126a^5 = 2 \times 84a^6$
$a = \frac{3}{4}$</td><td>1
1
1
1</td><td></td></tr>
<tr><td>6(c)(i)</td><td>P(at most one even) = P(none even) + P(one even)
$= p^6 + {}^6C_1 p^5(1-p)$
$= p^6 + 6p^5 - 6p^6$
$= 6p^5 - 5p^6$</td><td>1
1</td><td></td></tr>
<tr><td>6(c)(ii)</td><td>For the product to be odd – all scores would be odd.
P(all odd) = $p^6 \to P$(even product) = $1 - p^6$</td><td>1
1</td><td></td></tr>
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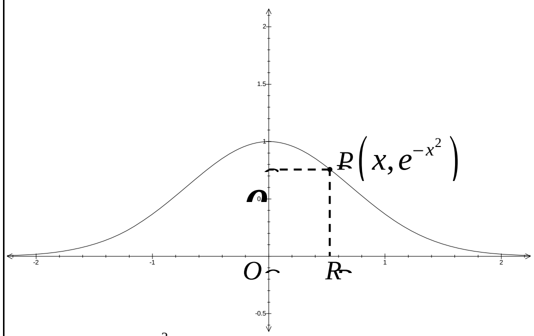
<tr><td>7</td><td>


$A = x \times e^{-x^2}$
$A' = e^{-x^2}.1 + x.-2xe^{-x^2}$
$\quad = e^{-x^2}\left(1-2x^2\right)$
Maximum area occurs when $A'=0$;
$e^{-x^2}\left(1-2x^2\right)=0$
$e^{-x^2}=0, \qquad 1-2x^2=0$
no solution $\qquad x=\frac{1}{\sqrt{2}}$ (since P is in the 1st quadrant)
<table>
<tr><td>x</td><td>0</td><td>1/√2</td><td>1</td></tr>
<tr><td>A’</td><td>+</td><td>0</td><td>-</td></tr>
</table>
/ — \
So this is a maximum area.
$A=\frac{1}{\sqrt{2}}e^{-\left(\frac{1}{\sqrt{2}}\right)^2}$
$\quad =\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}=\frac{1}{\sqrt{2}}\frac{1}{\sqrt{e}}=\frac{1}{\sqrt{2e}}$
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1</td><td></td></tr>
<tr><td>7(b)(i)</td><td>
$\sin x+\cos x = A\sin\left(x+\alpha\right)$
$\sin x+\cos x = A\sin x\cos\alpha + A\cos x\sin\alpha$
$A\cos\alpha=1, A\sin\alpha=1 \to \frac{A\sin\alpha}{A\cos\alpha}=1 \to \tan\alpha=1\to\alpha=\frac{\pi}{4}$
$A=\sqrt{1^2+1^2}=\sqrt{2}\to\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right).$
</td><td>1
1</td><td></td></tr>
<tr><td>7(b)(ii)</td><td>
$y=e^x\sin x\to y'=\sin x.e^x+e^x.\cos x=e^x\left(\cos x+\sin x\right)$
but $\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$
$y'=e^x\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=\sqrt{2}e^x\sin\left(x+\frac{\pi}{4}\right)$
</td><td>1
1</td><td></td></tr>
</table>

7(b)(iii)	RTP $\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin\left(x + \frac{n\pi}{4}\right)$. when n = 1: $\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin\left(x + \frac{n\pi}{4}\right) \to \frac{dy}{dx} = \left(\sqrt{2}\right)^1 e^x \sin\left(x + \frac{1\pi}{4}\right)$ true from (ii) suppose it is true for some value of n say k. $\frac{d^k y}{dx^k} = \left(\sqrt{2}\right)^k e^x \sin\left(x + \frac{k\pi}{4}\right)$. Prove it is true for $n = k + 1$ RTP $\frac{d^{k+1} y}{dx^{k+1}} = \left(\sqrt{2}\right)^{k+1} e^x \sin\left(x + \frac{(k+1)\pi}{4}\right)$. $\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\frac{d^k y}{dx^k} = \frac{d}{dx}\left(\sqrt{2}\right)^k e^x \sin\left(x + \frac{k\pi}{4}\right)$. $= \left(\sqrt{2}\right)^k \left(\sin\left(x + \frac{k\pi}{4}\right)e^x + e^x \cos\left(x + \frac{k\pi}{4}\right)\right)$ $= \left(\sqrt{2}\right)^k e^x \left(\sin\left(x + \frac{k\pi}{4}\right) + \cos\left(x + \frac{k\pi}{4}\right)\right)$ $= \left(\sqrt{2}\right)^k e^x \sqrt{2} \sin\left(x + \frac{k\pi}{4} + \frac{\pi}{4}\right)$ from (i) $= \left(\sqrt{2}\right)^{k+1} e^x \sin\left(x + \frac{(k+1)\pi}{4}\right)$ proved by induction.	1 1 1	